A generalization of the Birthday problem and the chromatic polynomial

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Abstract

The birthday paradox states that there is at least 50% chance that some two out of twenty-three randomly chosen people will share the same birth date. The calculation for this problem assumes that all birth dates are equally likely. We consider the following two modifications of this question. What if the distribution of birth dates is non-uniform? Further, what if we focus on birthdays shared by two friends rather than any two people? In this paper we present our results and conjectures in this generalized setting. We will also show how these results are related to the Stanley-Stembridge poset chain chromatic conjecture and the 'shameful conjecture', two famous conjectures in combinatorics.

1   Introduction

The Birthday problem is a classical and well-studied problem in elementary probability. There is a vast literature on this problem and it's generalizations and their applications; for example see [24], [32], [8], [16], [17], [22]. The birthday problem asks for the minimum number n of birthdays that we need to sample independently so that the probability that all of them are distinct is small (say less than 50%). The well known answer to this question is 23. To see this, suppose we have n people each having one of q possible birthdays distributed uniformly and independently. The probability that everybody has a distinct birthday is:

$$\prod_{i=1}^{n-1}\left(1-\frac{i}{q}\right). \tag{1.1}$$

For $q = 365$ this probability goes below 0.5 for the first time when $n = 23$. This has led to a popular demonstration in an introductory probability courses: In a class of about 25 to 30 students, birthdays are called out and it is observed very often that some two students share a common birthday.

One wonders though if it is accurate to assume that all birthdays occur with equal probability. There are more induced births during the weekdays than on weekends because of ready availability of staff. There may be fluctuations in birthrates during different seasons. Does this affect the probability of two students sharing the same birthday? If so, does the probability increase or decrease? It is known (for example, see [15], [4], [26]) that the probability of matching birthdays increases if the distribution of birthdays is not uniform. To see this, let $\mathbf{p} = (p_1, \ldots, p_q)$ be the distribution on the q birth dates and let $P_n(p_1, \ldots, p_q)$ denote the probability that no two people

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share the same birthday under this distribution. Then,

$$P_n(p_1, \ldots, p_q) = n! \sum_{i_1 < \ldots < i_n} (p_{i_1} \ldots p_{i_n}). \tag{1.2}$$

By a classical theorem of Muirhead [25] this is a concave symmetric function of the $p_i's$. Hence,

$$P_n(p_1, \ldots, p_q) \leq P_n\left(\frac{1}{q}, \ldots \frac{1}{q}\right). \tag{1.3}$$

Thus, in this case the uniform distribution is the worst case distribution i.e. the probability of all distinct birth dates is maximixed when the birthdates are uniformly distributed. Hence the answer 23 works even though the actual distribution of birth dates is unknown.

Further generalizing the situation, what happens if instead of all distinct birth dates we just want all pairs of friends to have distinct birth dates? We construct a friendship graph G as follows: there is a vertex corresponding to each person and an edge between two if and only if they are friends. Now replacing birth dates by q colors we get the following graph theory problem. Consider a graph G on n vertices. Suppose the vertices are colored at random with q colors occurring with probabilities $p_1 \cdots p_q$. We say that a coloring of a graph is a proper coloring if no edge is monochromatic. Let $P_G(p_1, \ldots, p_q)$ denote the probability that the random coloring thus obtained is a proper coloring. In this setting the Birthday Problem asks for the smallest n such that,

$$P_{K_n}\left(\frac{1}{q}, \ldots, \frac{1}{q}\right) \leq \frac{1}{2}. \tag{1.4}$$

In the general setting the distribution $\mathbf{p} = (p_1, \ldots, p_q)$ need not be uniform. Also G can be any underlying graph which we call the friendship graph. Equation (1.3) tells us that $P_{K_n}(p_1, \ldots, p_q)$ is maximized if all the colors occur with probability $p_i = 1/n$, where K_n denotes the complete graph on n vertices. A natural question to ask is if this is true for all underlying graphs G, i.e.

$$\text{Is } P_G(1/q, \ldots, 1/q) \geq P_G(p_1, \ldots, p_q) \text{ for all graphs } G? \tag{1.5}$$

The answer to this question is negative. In Section 2 we present two families of examples showing this. Thus the uniform distribution does not maximize $P_G(p_1, \ldots p_q)$ for all underlying friendship graphs G. We can ask two questions now:

- Is there a class of graphs where it does?

- Can we say something for general graphs?

The main results in our paper answer these questions:

Theorem 1.1. *If G is claw-free then $P_G(p_1, \ldots p_q)$ is maximized when $p_1 = \cdots = p_q = 1/q$. In fact P_G is Schur-concave on the set of probability distributions $\mathbf{p} = (p_1, \ldots, p_q)$.*

Theorem 1.2. *If $G = (V, E)$ is a graph with maximum degree Δ, then for $q > 6.3 \times 10^5 \Delta^4$ we have,*

$$P_G\left(\frac{1}{q}, \ldots, \frac{1}{q}\right) \geq P_G(p_1, \ldots, p_q), \tag{1.6}$$

for any distribution $\mathbf{p} = (p_1, \ldots, p_q)$ on the colors.

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Theorem 1.3. *If $G = (V, E)$ is a graph with maximum degree Δ, then for $q > 400\Delta^{3/2}$ we have,*

$$P\left(\frac{1}{q-1}, \ldots, \frac{1}{q-1}\right) \leq P\left(\frac{1}{q}, \ldots, \frac{1}{q}\right) \tag{1.7}$$

Theorem 1.3 relates to the 'shameful conjecture' as explained in Section 4.2.

One can further generalize this problem to allow some monochromatic edges. Thus, let $P_G(k, p_1, \ldots, p_q)$ denote the probability that a random coloring of G as above leads to at most k monochromatic edges. We will say that a graph G is 'P-uniform' if for all q, k, probability $P_G(k, p_1, \ldots, p_q)$ is maximized by the uniform distribution. Even though for $k = 0$ indeed P_G is maximized by the uniform distriubtion for claw-free graphs, it turns out that claw-free graphs are not in general P-uniform as shown by the following example. In Section 7 we use combinatorial arguments to show that the uniform distribution maximizes $P_G(k, p_1, \ldots, p_q)$ for all k when G is a complete graph or a cycle.
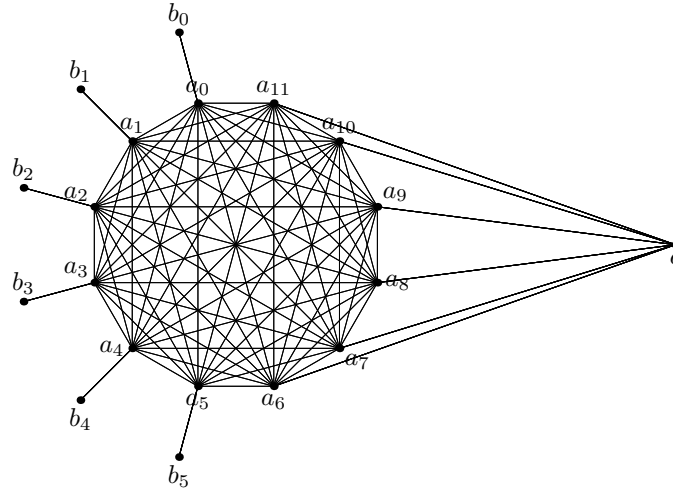

Figure 1: This figure shows claw-free graph G on 19 vertices for which $P_G(30, p_1, p_2)$ is no maximized by the uniform distribution.

Example: Consider G on 19 vertices $\{a_0, \cdots, a_{11}, b_0, \cdots, b_5, c\}$, as shown in Figure 1. The graphs induced on $\{a_0, \cdots, a_{11}\}$ and $\{a_6, \cdots, a_11, c\}$ are complete graphs respectively. There are also 6 edges (a_i, b_i) for $0 \leq i \leq 5$. This graph is claw-free. Now we show that $P_G(30, 0.5, 0.5) < P_G(30, 0.4, 0.6)$ for G. Note that for any 2-coloring of G, the complete graph on $\{a_0, \cdots, a_{11}\}$ will give rise to at least $2 \times \binom{6}{2} = 30$ monochromatic edges. Exactly 30 monochromatic edges are achieved if vertices $\{a_0, \cdots, a_5, c\}$ are colored c_1 and $\{a_6, \cdots, a_{11}, b_0, \cdots b_5, c\}$ are colored c_2 or vice-versa. Thus,

$$P_G(30, p_1, p_2) = p_1^7 p_2^{12} + p_2^7 p_1^{12}.$$

Thus,

$$P_G(30, 0.5, 0.5) = \frac{1}{2^18} < 3.9 \times 10^{-6} < 4 \times 10^{-6} < P_G(30, 0.4, 0.6).$$

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Unfortunately we do not know much more about about general graphs in this situation. Our guess is that as in the earlier case the uniform distribution should maximize P_G as q grows large enough. But at the moment we do not know how to prove this.

1.1 Graph coloring and chromatic polynomials

Let $G = (V, E)$ be a finite simple graph on n vertices. We say that a function $\alpha : V \to \{1, \ldots, q\}$ is a q-coloring of G if for each edge (u, v) of G we have $\alpha(u) \neq \alpha(v)$. Let $\chi_G(q)$ be the number of q-colorings of G. In general given a graph G it is difficult to say whether it has a q-coloring or not, and hence also difficult to count the exactly number of q-colorings. Using inclusion exclusion we see that P_G is in fact a polynomial known as the *chromatic polynomial*:

$$\chi_G(q) = \sum_{E' \subset E} (q)^{C(E')} (-1)^{|E'|}, \tag{1.8}$$

where $C(E')$ denotes the number of connected components in E'.

We note that $P_G(p_1, \ldots, p_q)$ can also be written as a polynomial of $p_1, \ldots, p_q$ in a similar manner:

$$P_G(p_1, \ldots, p_q) = \sum_{E' \subseteq E} (-1)^{|E'|} \prod_{\substack{\gamma \subset E' \\ \gamma \text{ connected.}}} (p_1^{|\gamma|} + \ldots + p_q^{|\gamma|}), \tag{1.9}$$

where the sum goes over all edges $E' \subseteq$ of the edges E, and the product is over all connected components of (V, E'). By $|\gamma|$ we denote the number of edges in γ. Note that the two polynomials are related to each other by the following equality:

$$P_G\left(\frac{1}{q}, \ldots, \frac{1}{q}\right) = \frac{\chi_G(q)}{q^n}. \tag{1.10}$$

Due to this similarity the study of $P_G(p_1, \ldots, p_q)$ is similar to the study of the chromatic polynomial $\chi_G(q)$. This is good because the chromatic polynomial is a very well-studied object. It was introduced by G. Birkhoff [3] as an approach for solving the four color problem. It was also generalized by Whitney and Tutte to the Tutte polynomial [33], [34] which has connections with the Potts model from statistical physics. The literature on chromatic polynomials is vast and we refer the reader to [28], [19] for excellent surveys. For the purposes of this paper we will be interested in the study of the roots of the chromatic polynomial [7], [6], [30], [5]. We will also show how our work is related to the study of the symmetric function generalization of the chromatic polynomial due to R.Stanley [31] and a chromatic polynomial inequality due to F.M.Dong [18].

The structure of the remaining paper is as follows. In Section 2 we provide examples showing that the answer to question 1.5 is negative in general. In Sections 3 and 4 we give relevant definitions and some motivation for the above theorems. In these sections we also show how these results are connected to the Stanley-Stembridge poset chain conjecture and the 'shameful conjecture' (now proved by F.M.Dong). In Sections 5 and 6 we give proofs of Theorems 1.1 and 1.2.

2 Two examples

Here we present two examples showing that the answer to question 1.5 is negative in general. The first example is due to Geir Helleloid.

Example 1 (Geir Helleloid): Consider the 'star graph' $K_{1,4}$ colored with two colors c_1, c_2 with respective probabilities p_1, p_2. Here $P(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2^4}$. On the other hand $P(\frac{1}{5}, \frac{4}{5}) = \frac{4^4}{5^5} + \frac{4}{5^5} > \frac{1}{2^4}$. In general if $G = K_{1,n}$ for $n \geq 4$, then,

$$P_G\left(\frac{1}{2}, \frac{1}{2}\right) < P_G\left(\frac{1}{n+1}, \frac{n}{n+1}\right). \tag{2.1}$$

Note that as we increase q the situation changes. In fact we will show in Section 4.1 that for star graphs $G = K_{1,n}$ the probability P_G is indeed maximized by the uniform distribution when $q \geq n$.
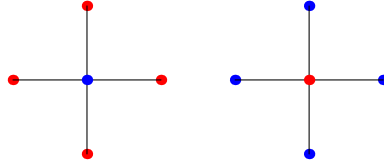

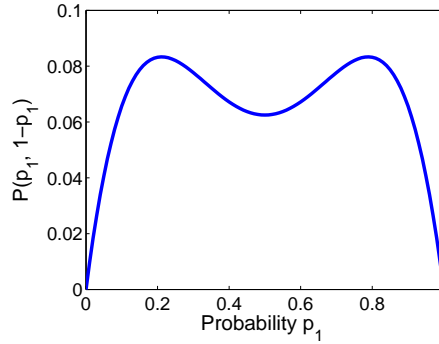Figure 2: Four star and it's two proper colorings with two colors.

Figure 3: Above is a plot of $P_{K_{1,4}}(p_1, 1 - p_1)$ against p_1. We see that $P_{K_{1,4}}(p_1, 1 - p_1)$ is maximized at $1/5$ and $4/5$.

Example 2: We now consider the regular rooted tree of degree 3 (the root has degree 2). Let T_k denote such a tree with depth $2k$. With two colors there are only two ways of coloring the tree without having monochromatic edges: Color all nodes at even numbered layers in one color and the nodes at odd layers with the other color. Hence

$$P(p_1, p_2) = p_1^{N_1} p_2^{N_2} + p_2^{N_1} p_1^{N_2} \tag{2.2}$$

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where $N_1 = (4^k - 1)/3$, the total number of nodes in the even layers and $N_2 = 2(4^k - 1)/3 = 2N_1$ is the total number of nodes in the odd layers. In particular,

$$P\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2^{3N_1}} \tag{2.3}$$

and,

$$
\begin{aligned}
&P\left(\frac{1}{2} - \frac{1}{2N_1}, \frac{1}{2} + \frac{1}{2N_1}\right) \\
&= \left(\frac{1}{2} - \frac{1}{2N_1}\right)^{N_1} \left(\frac{1}{2} + \frac{1}{2N_1}\right)^{2N_1} + \left(\frac{1}{2} + \frac{1}{2N_1}\right)^{N_1} \left(\frac{1}{2} - \frac{1}{2N_1}\right)^{2N_1} \\
&= \frac{1}{2^{3N_1}} \left(1 - \frac{1}{N_1}\right)^{N_1} \left(1 + \frac{1}{N_1}\right)^{N_1} \left[\left(1 + \frac{1}{N_1}\right)^{N_1} + \left(1 - \frac{1}{N_1}\right)^{N_1}\right]
\end{aligned} \tag{2.4}
$$

which is asymptotically equal to

$$\frac{1}{2^{3N_1}} \left(e + \frac{1}{e}\right) > \frac{1}{2^{3N_1}} \tag{2.5}$$

in the limit as $k \to \infty$ and hence $N_1 \to \infty$.

Hence for large values of k we have that $P_0((\frac{1}{2} - \frac{1}{2N_1}), (\frac{1}{2} + \frac{1}{2N_1})) > P_0(\frac{1}{2}, \frac{1}{2})$.
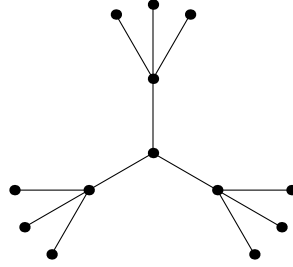

Figure 4: 3-ary tree

3   Clawfree graphs

Definition 3.1. *A* **claw** *is the bipartite graph $K_{1,3}$. We say that a graph has an* **induced claw** *if it has a subgraph on 4 vertices $\{a, b, c, d\}$ such that the only edges between these four vertices are $\{ab, ac, ad\}$. A graph is said to be* **claw-free** *if it does not have any induced subgraphs isomorphic to $K_{1,3}$.*

Examples of clawfree graphs include complete graphs, cycles, complements of triangle-free graphs, line graphs, etc.

Claw-free graphs are a very well-studied class of graphs. P. Seymour and M.Chudnovsky gave a complete classification of these graphs in [9], [10], [11], [12], [13], [14]. They showed that claw-free graphs can be obtained by composing graphs from a few basic classes such that line graphs,
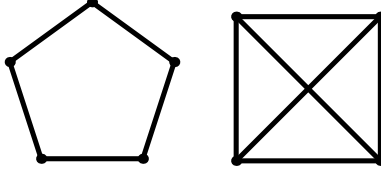
Figure 5: A claw, $K_{1,3}$.

Figure 6: Examples of claw-free graphs

proper circular arc graphs, etc. Many algorithms have also been well-studied for claw-free graphs. For example, algorithms for finding the maximum independent sets and computing independence polynomials for claw-free graphs are studied in [29], [2]. This was also the first class of graphs for which the perfect graph conjecture was proved. For a nice survey about claw-free graphs see [20].

We now recall the definition of partial ordering by majorization for vectors of real numbers. Given v, w in $\mathbb{R}^d$, let v_i and w_i be the coordinates of v and w written in decreasing order. For example, if $v = (1, 3, 5)$ and $w = (4, 4, 1)$, then $v_1 = 5, v_2 = 3, v_3 = 1$ and $w_1 = 4, w_2 = 4, w_3 = 1$.

Definition 3.2. *We say that $v \succeq w$ if*

$$\sum v_i \geq \sum w_i.$$

So, in the above example, $v \succeq w$.

Definition 3.3. *We say that a function $f : \mathbb{R}^d \to R$ is* **Schur concave** *if,*

$$v \succeq w \text{ implies } f(v) \leq f(w).$$

We note that the function f such that $f(v) = \prod v_i$ is Schur concave. In particular, $v \succeq (1/d \cdots 1/d)$ and hence $f(1/d \cdots 1/d) \geq f(v)$ for all v such that $v_i \geq 0$ and $\sum v_i = 1$. A lot more about majorization and schur concavity can be found in [23].

Now that we have all the definitions we restate our result for claw-free graphs:

Theorem. *(1.1) If G is claw-free then $P_G(p_1, \ldots p_q)$ is maximized when $p_1 = \cdots = p_q = 1/q$. In fact P_G is Schur-concave on the set of probability distributions $\boldsymbol{p} = (p_1, \ldots, p_q)$.*

Note that there could be other families of graphs with this property.

3.0.1 Connection to the Stanley Stembridge poset chain conjecture

Persi Diaconis conjectured theorem 6 based on a conjecture of Richard Stanley and John Stembridge which states:

Conjecture 3.4. *Let G be the incomparability graph of a $(3+1)$-free poset. The symmetric function generalization of the chromatic polynomial $\chi_G(x_1, \ldots, x_q)$ is e-positive.*

We now define some of the terms and explain the connection to the above theorem.

Definition 3.5. *Let G be a graph with chromatic number $\chi(G) \leq q$. A function $\sigma : V(G) \to [q]$ is called a* **coloring** *of G if $\sigma(u) \neq \sigma(v)$ for all adjacent nodes u, v. Then the* **symmetric function generalization** *of the chromatic polynomial as defined by R. Stanley [31] is:*

$$\chi_G(x_1, \ldots, x_q) := \sum_{\sigma:V(G) \to [q]} \prod_{v \in V(G)} x_{\sigma(v)}.$$

For example let H be an edge, then $\chi_H(x_1, x_2, x_3) = 2(x_1 x_2 + x_2 x_3 + x_3 x_1)$. An important observation of Persi Diaconis that we will use is that for all probability distributions $p = (p_1, \ldots, p_q)$ one has the equality,

$$\chi_G(p_1, \ldots, p_q) = P_G(p_1, \ldots, p_q). \tag{3.1}$$

Definition 3.6. *A* **poset** *is a set P with a binary relation "$\leq$" such that it is reflexive ($a \leq a$), antisymmetric (if $a \leq b$ and $b \leq a$ then $a = b$) and transitive (if $a \leq b$ and $b \leq c$ then $a \leq c$).*

Definition 3.7. *The* **incomparability graph** *of a poset P is a graph with the elements of P as nodes. There is an edge distinct a and b if they are incomparable. That is, if there is no directed path between a and b.*
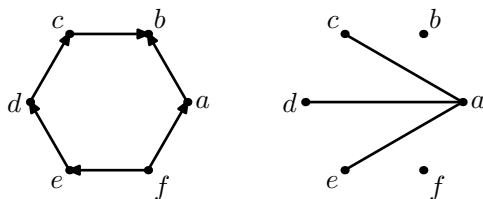

Figure 7: On the left is an example of a poset on 6 elements and on the right is it's incomparability graph.

Definition 3.8. *A $(3+1)$-free poset is a poset such that there is no set of four elements a, b, c, d such that a, b, c are mutually comparable but all three are incomparable with d.*

Note that the incomparability graph of $(3 + 1)$-free graphs is claw-free. The poset shown in figure 3.0.1 is not $(3 + 1)$-free.

Definition 3.9. *The* **elementary symmetric polynomials** *are defined as:*

$$e_k(x_1, \ldots, x_q) = \sum_{i_1 < \ldots < i_k} x_{i_1} \ldots x_{i_k}.$$

Note that every symmetric polynomial can be written as a polynomial of the elementary symmetric polynomials.

Definition 3.10. *A symmetric polynomial $p(x_1, \ldots, x_q)$ is said to be* **e-positive** *if it can be written as a positive linear combination of products of the elementary symmetric polynomials.*

Example: $f(x_1, x_2) = x_1^2 + 4x_1 x_2 + x_2^2 = e_1^2 + 2e_2$ is e-positive but $g(x_1, x_2) = x_1^2 + x_2^2 = e_1^2 - 2e_2$ is not e-positive.

3.0.2 Connection

Now we are ready to state the connection between the Stanley-Stembridge poset-chain conjecture and theorem 1.1. As noted above the incomparability graph G of a $(3 + 1)$- free poset is claw-free. Hence if the conjecture is true then $\chi_G(x_1, \ldots, x_q)$ is e-positive. Since the the elementary symmetric polynomials are schur-concave, it follows that their restriction to the set of probability distributions is unimodal with a maximum at $(1/q, \ldots, 1/q)$. Hence $P_G(p_1, \ldots, p_q) = \chi_G(p_1, \ldots, p_q)$ also attains a maximum at $(1/q, \ldots, 1/q)$. Thus the Stanley-Stembridge poset-chain conjecture implies theorem 1.1 for the subset of claw-free graphs which as incomparability graphs of $(3+1)$-free posets. It was because of this implication that Persi Diaconis first conjectured theorem 1.1.

We note that the Stanley chromatic polynomial is not *e-positive* for all claw-free graphs. An example of such a claw-free graph is provided in [31], figure 5.

A weaker version of Richard Stanley's conjecture states that the symmetric function is *s-positive* for claw-free graphs. This was proved by V. Gasharov in [21] and later by S. Assaf. Unfortunately this does not prove that $P_G(p_1, \ldots, p_q)$ is a schur-concave function since all Schur functions are not Schur concave. For example the Schur function $s_{5,1}(x, y) = x^5 y + x^4 y^2 + x^3 y^3 + x^2 y^4 + xy^5$ is not Schur concave (example due to Geir Helleloid).

4 General graphs

The example of the star graphs showed that question 1.5 does not have a positive answer in general. In that example we restricted our attention to the case $q = 2$. What happens if we let q increase? In figure 4 we show the contour map of $P_S(p_1, p_2, p_3)$. We see that towards the edges of the picture the contour lines do not enclose a convex region. But as we move inwards contour lines with higher values we see that the region enclosed comes closer to being convex. This suggests the possibility that increasing the number of colors gives a positive answer to question 1.5. The question is how large should q be for this to hold true. One might think that the answer is true asymptotically in the calue of q or that q must grow with the size of the graph. In fact, as stated in theorem 1.2 we show that q needs to grow as a polynomial of the highest degree of the graph.

In the next section we prove this result for star graphs. This proof gives us some insight into the proof for the general case.
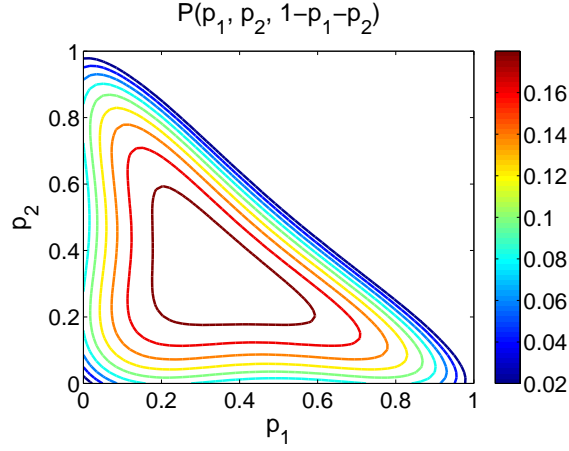
$$P(p_1, p_2, 1-p_1-p_2)$$

Figure 8: This figure shows the contour map for $P_G(p_1, p_2, 1 - p_1 - p_2)$ where the graph G is the 4-star.

4.1 Star graphs

Theorem 4.1. *For the star graph $G = K_{1,n}$ and $q > n$ we have,*

$$P_G(p_1, \ldots, p_q) \leq P_G\left(\frac{1}{q}, \ldots, \frac{1}{q}\right). \tag{4.1}$$

Proof. Given the star graph and colors as above, the probability that a random coloring gives rise to a proper coloring is:

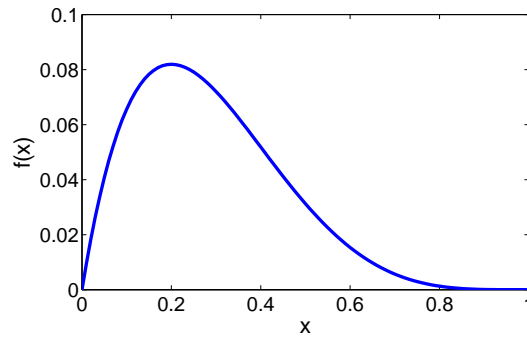$$P_G(p_1, \ldots, p_q) = \sum_{i=1}^{q} p_i(1 - p_i)^n. \tag{4.2}$$

Figure 9: This is a plot of $f(x) = x(1 - x)^4$ against x. We see that f is maximized at 0.2 and is concave on $[0, 0.4]$.

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Note that the function $f(x) = x(1-x)^n$ is unimodal for $0 \le x \le 1$. In fact, it is concave on $[0, \frac{2}{n+1}]$ and convex on $[\frac{2}{n+1}, 1]$. The function has a unique maxima at $\frac{1}{n+1}$ on the interval $[0, 1]$. Let

$$\Omega = \{x_1 \dots x_q | x_i \ge 0, x_1 + \dots x_q = 1\}.$$

We wish to show that P_G has a maximum at $(\frac{1}{q}, \dots, \frac{1}{q})$, on Ω. Let

$$\Theta = \{(x_1, \dots, x_q) \in \Omega | x_i \le \frac{2}{n+1} \text{ for all } i\}.$$

Then by the unimodality and concavity of $x(1-x)^n$ on $[0, \frac{2}{n+1}]$, it follows that P_G has a maxima at $(\frac{1}{q}, \dots, \frac{1}{q})$, on Θ. Now suppose $(x_1, \dots, x_q) \in \Omega$ is such that $x_i > \frac{2}{n+1}$ for some i. Then there is also an x_j such that $x_j < \frac{1}{n+1}$. Then replacing x_i by $x_i + x_j - \frac{1}{n+1}$ and x_j by $\frac{1}{n+1}$ increases the value of P_G. Continuing thus, we can get to a point in Θ where the value of P_G will be strictly greater than the value of P_G at the point outside Θ where we started. This together with the earlier fact proves that P_G has a maximum at $(\frac{1}{q}, \dots, \frac{1}{q})$, on Ω. $\square$

4.2   Shameful conjecture

The chromatic polynomial is a very well-studied subject. Even though evaluating chromatic polynomial exactly is a difficult problem in general, many of it's properties have been studied extensively.

D.Welsh and Bartels [1] studied the expected number of colors used in a q-coloring on graph G (under uniform distribution on all colorings). The following result relates this average μ to the chromatic number of the graph:

Theorem 4.2. *(Bartels, Welsh [1], G. Rote) If G is a graph on n vertices,*

$$\mu(G) = n \left(1 - \frac{P_G(n-1)}{P_G(n)} \right). \tag{4.3}$$

Bartel and Welsh conjectured that out of all graphs G on n vertices $\mu(G)$ is minimized when the graph has no edges. This conjecture came to be known as the 'shameful conjecture' and was proved recently by F.M.Dong [18]. The formal statement of the theorem is:

Theorem 4.3. *(F.M. Dong [18])*

$$n \left(1 - \frac{(n-1)^n}{n^n} \right) \le n \left(1 - \frac{P_G(n-1)}{P_G(n)} \right)$$

$$\tag{4.4}$$

$$i.e. \ \frac{P_G(n-1)}{(n-1)^n} \le \frac{P_G(n)}{n^n}.$$

The equation can be rewritten in our notation as:

$$P_G \left(\frac{1}{q-1}, \dots, \frac{1}{q-1} \right) \le P_G \left(\frac{1}{q}, \dots, \frac{1}{q} \right). \tag{4.5}$$

Note that, $P_G(1/r, \dots, 1/r)$ is the probability that a uniformly chosen function $f : V(G) \to \{1, \dots r\}$ is a coloring of G. Thus the 'shameful conjecture' can be interpreted as saying that there

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is a higher probability that a random map $f : V(G) \to \{1, \ldots n\}$ is a coloring of G than that a random map $f : V(G) \to \{1, \ldots n-1\}$ is a coloring of G. This seems natural and one might even think that this should hold for any q, i.e.

$$P_G\left(\frac{1}{q-1}, \ldots, \frac{1}{q-1}\right) \leq P_G\left(\frac{1}{q}, \ldots, \frac{1}{q}\right). \tag{4.6}$$

But as shown by Colin McDiarmid [1] the above statement is not true in general. $G = K_{n,n}$ with $q = 3$ provides a counter example.

Note that theorem 1.3 shows that even though equation (4.6) is false in general it holds true when $q > C\Delta^2$ where C is an explicit constant independent of the graph and Δ is the highest degree of the graph. We also show that equation (4.6) is always true for a class of graphs called claw-free graphs. Putting together theorem 1.3 and the theorem of F.M.Dong [18] we get the following strengthening of their result:

Theorem 4.4. *For a graph G with maximum degree Δ and $q > \min\{n-1, 400\Delta^{3/2}\}$ we have,*

$$\frac{P_G(q)}{(q)^n} \leq \frac{P_G(q+1)}{(q+1)^n}. \tag{4.7}$$

5   Proof for Claw-free graphs
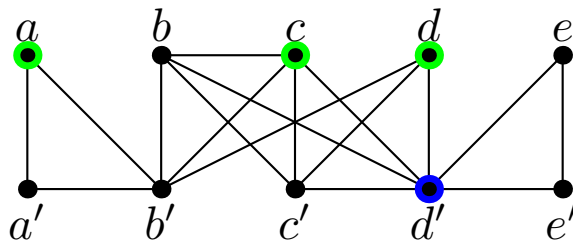
Here we shall prove theorem 1.1

Figure 10: This figure shows a claw-free graph G colored with colors 3,4 (here blue and green). The graph induced on the remaining vertices is a union of three disjoint chains $\{a'\}$, $\{b'bc'\}$ and $\{ee'\}$.

Proof. Suppose G is a disjoint union of connected claw-free graphs $G_1, \ldots, G_l$. Then note that,

$$P_G(p_1, \ldots, p_q) = \prod_{i=1}^{l} P_{G_i}(p_1, \ldots, p_q). \tag{5.1}$$

If each term in the product on the right hand side is non-negative, schur-concave then the left hand side is also schur-concave. Hence, it suffices to consider G connected. Now suppose we start with a distribution $p = (p_1, \ldots, p_q)$ on the colors $1, \ldots, n$. Fix p_i for $i \geq 3$. Let $H \subseteq G$ be a subgraph of G. We denote by C_H the set of all colorings of H with colors 3 to q. Let H' be the subgraph of G induced by the vertices of G not in H. Let $N(H, a_3, \ldots, a_q)$ denote the number of proper

colorings of H with a_i vertices colored with color i. Note that $N(H, a_3, \ldots, a_r)$ is independent of the $p_i's$. Then, by Bayes' rule we have,

$$
\begin{aligned}
&P_G(p_1, \ldots, p_q) \\
&= \sum_{H \subseteq G} \sum_{(a_3, \ldots, a_q)} N(H, a_3, \ldots, a_q) \prod_{i=3}^r p_i^{a_i} \times P_{H'} \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \right) \times (p_1 + p_2)^{|V(H')|},
\end{aligned}
\tag{5.2}
$$

where H' is the subgraph induced by the remaining vertices. To show that P_G is schur-concave it suffices to show that

$$
P_{H'} \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \right) (p_1 + p_2)^{|V(H')|}
\tag{5.3}
$$

is maximized when $p_1 = p_2$. To see this note that H' is also claw-free since removing vertices keeps a claw-free graph claw-free. If H' is not bipartite then it cannot have a proper coloring with 2 colors. Hence for the above term to be non-zero H' must be a claw-free bipartite graph. The only connected claw-free bipartite graphs are cycles of even length and paths. To see this, suppose $V(H')$ is partitioned into sets A, B such that there are no edges lying entirely inside A or B. So for any $v \in A$ all it's neighbors lie in B. Suppose v has three neighbors v_1, v_2, v_3. Since they are all in B there are no edges between them. This leads to a claw on v, v_1, v_2, v_3 and contadicts the fact that H' is claw-free. Hence the maximum degree of H' is 2, thus implying that H' is a disjoint union of cycles and paths. Further since H' is bipartite the cycles can only have even length. Thus,

$$
P_{H'} \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \right) (p_1 + p_2)^{|V(H')|}
$$

$$
\tag{5.4}
$$

$$
= 2(p_1 p_2)^k \text{ or } p_1^k p_2^{k+1} + p_2^k p_1^{k+1},
$$

depending on whether H' has $2k$ or $2k+1$ vertices. In both cases this is maximized when $p_1 = p_2$. So,

$$
P(p_1, \ldots, p_q) \leq P \left(\frac{p_1 + p_2}{2}, \frac{p_1 + p_2}{2}, p_3, \ldots, p_q \right)
\tag{5.5}
$$

and by symmetry,

$$
P(p_1, \ldots, p_q) \leq P \left(p_1, \ldots, p_{i-1}, \frac{p_i + p_j}{2}, p_{i+1}, \ldots, p_{j-1}, \frac{p_i + p_j}{2}, p_{j+1}, \ldots, p_q \right),
\tag{5.6}
$$

which proves that P_G is schur concave. The above inequality attains equality if $p_i = p_j$. Further, since P_G is a continuous function on a compact space, a maximum is attained and by the above inequality it must be attained when $p_1 = \cdots = p_q = 1/q$.

This completes the proof.

$\square$

6 Proof for general graphs

In this section we will prove theorem 1.2 and 1.3. We restate the theorems below:

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Theorem. *(1.2) If $G = (V, E)$ is a graph with maximum degree Δ, then for $q > 6.3 \times 10^5 \Delta^4$ we have,*

$$P_G(1/q, \ldots, 1/q) \geq P_G(p_1, \ldots, p_q), \tag{6.1}$$

for any distribution $\boldsymbol{p} = (p_1, \ldots, p_q)$ on the colors.

Theorem. *(1.3) If $G = (V, E)$ is a graph with maximum degree Δ, then for $q > 400\Delta^{3/2}$ we have,*

$$P\left(\frac{1}{q-1}, \ldots, \frac{1}{q-1}\right) \leq P\left(\frac{1}{q}, \ldots, \frac{1}{q}\right) \tag{6.2}$$

Proof. (Proof of 1.2) As in the case of the star graph, the proof in the general case has two steps. The first step is to show that if any p_i is much larger than $1/q$ then, $P_G(1/q, \ldots, 1/q) \geq P_G(p_1, \ldots, p_q)$, more precisely,

Theorem 6.1. *If $p_i \geq 2\sqrt{\dfrac{\Delta}{q}}$ for some i, then $P(p_1, \ldots, p_q) \leq P(1/q, \ldots, 1/q)$.*

The next step is to show that when all the $p_i's$ are close to $1/q$ then P_G is log-concave for large enough q:

Theorem 6.2. *If $q > 6.3 \times 10^5 \Delta^4$, then $P_G(p_1, \ldots, p_q)$ is log-concave in the region $\Omega = \left\{(p_1, \ldots, p_q) | p_i \leq 2\sqrt{\frac{\Delta}{q}}\right\}$. And hence P_G is maximized at $(1/q, \ldots, 1/q)$ on Ω.*

Theorem 6.1 and theorem 6.2 together prove theorem 1.2. We prove these theorems in sections 6.1 and 6.2. $\qquad\square$

Proof. (of theorem 1.3) Similar to the proof of theorem 6.2 we prove:

Theorem 6.3. *If $q > 400\Delta^{3/2}$, then $P_G(p_1, \ldots, p_q)$ is log-concave in the region $\Omega = \left\{(p_1, \ldots, p_q) | p_i \leq \frac{1}{q-1}\right\}$. And hence P_G is maximized at $(1/q, \ldots, 1/q)$ on Ω, in particular,*

$$P_G\left(\frac{1}{q}, \ldots, \frac{1}{q}\right) \geq P_G\left(\frac{1}{q-1}, \ldots, \frac{1}{q-1}, 0\right). \tag{6.3}$$

Theorem 6.3 is proved in section 6.2.

$\qquad\square$

6.1 Proof of theorem 6.1

Proof. Let $N = \chi_G(q)$ be the number of proper colorings of G using q colors. Suppose the vertices of G have degrees $d_1, \ldots, d_n$ respectively. Then $2|E| = \sum_{i=1}^{n} d_i$. Note that for $q > \Delta$,

$$\frac{N}{q^n} \geq \prod_{i \leq n}\left(\frac{q - d_i}{q}\right) \geq \left(\frac{q - \Delta}{q}\right)^{(\sum d_i)/\Delta} = \left(1 - \frac{\Delta}{q}\right)^{2|E|/\Delta}. \tag{6.4}$$

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The first inequality follows by coloring vertices in a fixed order. Vertex i can have any of $q - n_i \geq q - d_i$ colors, where n_i is the number neighbors of vertex i that have already been colored. To see the second inequality, note that for $1 \geq a \geq b \geq 0$ and $\epsilon \geq 0$ one has,

$$(1 - a - \epsilon)(1 - b + \epsilon) = 1 - a - b + ab - \epsilon(a - b) - \epsilon^2 \leq (1 - a)(1 - b). \tag{6.5}$$

This implies that $\log(\prod_{i \leq n}(1 - x_i))$ is schur-concave. Thus,

$$\prod_{i \leq n} \left(1 - \frac{d_i}{q}\right)^{\Delta} \geq \left(1 - \frac{\Delta}{q}\right)^{2|E|} \times 1^{n\Delta - 2|E|}, \tag{6.6}$$

since $(d_1, \ldots, d_1, \ldots, d_n, \ldots, d_n) \preceq (\Delta, \ldots, \Delta, 1, \ldots, 1)$ where the first vector has Δ co-ordinates that are d_i for each i and the second vector has $2|E|$ co-ordinates that are Δ and the rest are 1's. This gives the second inequality in 6.4.

Hence,

$$P(1/q, \ldots 1/q) = \frac{N}{q^n} \geq \left(1 - \frac{\Delta}{q}\right)^{2|E|/\Delta}. \tag{6.7}$$

Now since the maximum degree is Δ we can find a set $U \subset E$ of $|E|/2\Delta$ disjoint edges in G. Hence,

$$P(p_1, \ldots, p_q) \leq (1 - \sum p_i^2)^{|E|/2\Delta} \tag{6.8}$$

So now it suffices to prove that

$$(1 - \sum p_i^2)^{|E|/2\Delta} \leq \left(1 - \frac{\Delta}{q}\right)^{2|E|/\Delta}, \tag{6.9}$$

that is,

$$(1 - \sum p_i^2) \leq \left(1 - \frac{\Delta}{q}\right)^4. \tag{6.10}$$

Or, since

$$1 - \frac{4\Delta}{q} \leq \left(1 - \frac{\Delta}{q}\right)^4, \tag{6.11}$$

it suffices to prove that

$$(1 - \sum p_i^2) \leq 1 - \frac{4\Delta}{q}$$
$$\text{i.e. } \frac{4\Delta}{q} \leq \sum p_i^2. \tag{6.12}$$

This is true by the hypothesis and hence completes the proof. $\qquad\square$

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6.2 Proof for theorem 6.2, 6.3

For the proof of theorem (6.2) we will make extensive use of ideas and theorems due to A. Sokal [30] and C.Borgs [5]. The first hurdle is to get a nice combinatorial, inductive formula for P_G. As stated earlier, inclusion-exclusion gives:

$$P_G(p_1, \ldots, p_q) = \sum_{E' \subseteq E} (-1)^{|E'|} \prod_{\gamma: \text{connected}} (p_1^{|\gamma|} + \ldots + p_q^{|\gamma|}), \tag{6.13}$$

where the product is over all connected components γ of (V, E'). Recall that if $A = A_1 \cup \ldots \cup A_k$ is a union of events then inclusion exclusion says:

$$\text{Prob}(A) = \sum_{i \leq k} \text{Prob}(A_i) - \sum_{1 \leq i < j \leq k} \text{Prob}(A_i \cap A_j) + \ldots + (-1)^{k+1} \text{Prob}(A_1 \cap \ldots \cap A_k). \tag{6.14}$$

So, let A be the event that the coloring is not a proper coloring and let A_i denote the event that edge i is monochromatic (i.e. both end points have the same color). Then since $A = A_1 \cup \ldots \cup A_{|E|}$, and $P_G(p_1, \ldots, p_q) = 1 - \text{Prob}(A)$, we get,

$$\begin{aligned} P_G(p_1, \ldots, p_q) &= 1 - \sum_{\emptyset \neq E' \subseteq E} (-1)^{|E'|+1} \prod_{\substack{\gamma \subset E' \\ \gamma \text{ connected.}}} (p_1^{|\gamma|} + \ldots + p_q^{|\gamma|}) \\ &= \sum_{E' \subseteq E} (-1)^{|E'|} \prod_{\substack{\gamma \subset E' \\ \gamma \text{ connected.}}} (p_1^{|\gamma|} + \ldots + p_q^{|\gamma|}) \end{aligned} \tag{6.15}$$

where the sum goes over all subsets $E' \subseteq E$, and the product is over all connected components of (V, E'). By $|\gamma|$ we denote the number of edges in γ. Also note that the summand is 1 when $E' = \emptyset$.

Thus, we can think of P_G as a complex multivariate polynomial $P_G(z_1, \ldots, z_q)$. Now P_G can be rewritten by collecting together subsets E' of E that lead to connected components on the same set of vertices. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote the graph whose set of vertices is given by the set of connected subsets S of V such that $|S| \geq 2$. There is an edge between S_1 and S_2 if $S_1 \cap S_2 \neq \emptyset$. Then, P_G can be rewritten as:

$$\begin{aligned} P_G(z_1, \ldots, z_q) &= \sum_{\substack{\mathcal{W} \subseteq \mathcal{V} \\ \mathcal{W} \text{ independent}}} \prod_{S_i \in S} w(S_i) \\ \text{where } w(S) &= (z_1^{|S|} + \ldots + z_q^{|S|}) \sum_{\substack{\gamma \subseteq E, \\ (S, \gamma) \text{ connected}}} (-1)^{|\gamma|}, \end{aligned} \tag{6.16}$$

where the summand when $\mathcal{W} = \emptyset$ is 1.

One advantage of writing P in this form is that it can be decomposed nicely. Let $\mathcal{U} \subseteq \mathcal{V}$. We define:

$$P_{\mathcal{U}} = \sum_{\substack{\mathcal{W} \subseteq \mathcal{U} \\ \mathcal{W} \text{ independent}}} \prod_{S_i \in S} w(S_i). \tag{6.17}$$

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Let $\eta \in \mathcal{V}$, and let $\mathcal{V}' = \mathcal{V} - \{\eta\}$. Further let, $\mathcal{V}_0 = \mathcal{V} - N[\{\eta\}]$, where $N[x]$ denotes the set containing x and it's neighbors in $\mathcal{G}$. Then,

$$P_{\mathcal{V}} = P_{\mathcal{V}'} + w(\eta)P_{\mathcal{V}_0}. \tag{6.18}$$

Such a decomposition is useful for proving statements inductively. For example, it is used to prove Dobrushin's theorem which gives conditions under which functions which can be decomposed as above are non-zero. Applying a version of Dobrushin's theorem (as explained in section 6.4) gives us:

Theorem 6.4 (Proved in 6.2.1). *Let Δ be the highest degree of G and let $K = 7.963907$ be a constant. If $q > K^2\Delta^3$ then $|\log P_G(z_1, \ldots, z_q)| \leq 4|E|/5$ in the region*

$$\Omega = \left\{ (z_1, \ldots, z_q) \in \mathbb{C}^q : |z_1|^m + \ldots + |z_q|^m \leq \left(2\sqrt{\frac{\Delta}{q}} \right)^{m-1} \forall m \in \mathbb{Z}_+ \right\}.$$

At this point we will prove the following useful lemma:

Lemma 6.5. *Let,*

$$\Omega_1 = \left\{ (p_1, \ldots, p_q) : p_i \geq 0, p_1 + \ldots + p_q = 1, |p_i| \leq 2\sqrt{\frac{\Delta}{q}} \right\} \tag{6.19}$$

and,

$$\Omega = \left\{ (z_1, \ldots, z_q) \in \mathbb{C}^q : |z_1|^m + \ldots + |z_q|^m \leq \left(2\sqrt{\frac{\Delta}{q}} \right)^{m-1} \forall m \in \mathbb{Z}_+ \right\}, \tag{6.20}$$

as above. Then, $\Omega_1 \subset \Omega$.

Proof. Let,

$$\left\lfloor \sqrt{\frac{q}{4\Delta}} \right\rfloor = k \text{ and } a = 1 - 2k\sqrt{\frac{\Delta}{q}} \leq 2\sqrt{\frac{\Delta}{q}}. \tag{6.21}$$

Since Ω_1 is a symmetric convex polytope and $p_1^m + \ldots + p_q^m$ is a symmetric convex function it is maximized on the endpoints. Thus, $p_1^m + \ldots + p_q^m \leq k\left(2\sqrt{\frac{\Delta}{q}} \right)^m + a^m \leq \left(2\sqrt{\frac{\Delta}{q}} \right)^{m-1}$ since $a^m \leq ab^{m-1}$ for all $b \geq a \geq 0$. $\qquad\square$

Using inclusion-exclusion we obtained equation (6.16) for P_G. The following combinatorial identity is used to rewrite the equation. Let $S_1, \ldots, S_n$ be subsets of V and let $F(X, Y) = 0$ if X, Y are disjoint and -1 otherwise. Then,

$$\sum_{H \in G_N} \prod_{<ij> \in H} F(S_i, S_j) = \begin{cases} 0 \text{ if } S_1, \ldots, S_n \text{ are disjoint,} \\ -1 \text{ otherwise,} \end{cases} \tag{6.22}$$

where G_N is the set of all graphs on N vertices. To see this, note that the sum can be interpreted at $(1-1)^k$ where k is the number of pairs (S_i, S_j) that are not disjoint. An intersecting pair (S_i, S_j)

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contributes 1 to the product if $< ij >$ is not an edge in H else it contributes -1. A disjoint pair (S_i, S_j) contributes 0 to the sum. This gives the above identity.

Thus equation (6.16) can be re-written as,

$$P_G(z_1, \ldots, z_q) = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{S_1, \ldots, S_N} \prod_i w(S_i) \sum_{H \in G_N} \prod_{<ij> \in G} F(S_i, S_j). \qquad (6.23)$$

The term when $N = 0$ is defined to be 1.

Using the exponential formula one gets the Mayer expansion,

$$\log P_G(z_1, \ldots, z_q) = \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{S_1, \ldots, S_N \in \mathcal{V}} \prod_i w(S_i) \sum_{H \in C_N} \prod_{<ij> \in G} F(S_i, S_j). \qquad (6.24)$$

Here C_N is the set of all connected graphs on N vertices.

Let $\nu_{\mathbf{z}}(m) = z_1^m + \ldots + z_q^m$ for $2 \le m \le n$. The Mayer expansion is a power series of $w(S_i)$, and hence also of $\nu_{\mathbf{z}}(m)$ and the coefficients are independent of q. Theorem 6.4 tells us that $|\log P_G| \le 4|E|/5$ on the polydisc defined by $|\nu_{\mathbf{z}}(m)| \le \left(\sqrt{2\frac{\Delta}{q}}\right)^{m-1}$ whenever $q > K^2 \Delta^3$. This implies the convergence of the Mayer expansion of P_G in this region and Corollary 6.6 gives us the following bounds on it's coefficients:

Corollary 6.6 (Proved in 6.2.2). *Suppose we rewrite $\log P_G(p_1, \ldots, p_q)$ expressed as the power series of $\nu_{\mathbf{z}}(m)$'s is,*

$$\log P_G(z_1, \ldots, z_q) = -|E|w(2) + \sum_{M=3}^{\infty} \sum_{\substack{\alpha = (\alpha_1 \le \ldots, \le \alpha_s): \\ partition\ of\ M}}^{\alpha_i \ge 2} C_\alpha \prod_1^s \nu_{\mathbf{z}}(\alpha_i). \qquad (6.25)$$

Then,

$$|C_\alpha| \le 4|E| \times (K\Delta)^{M-s}/5, \ \ when\ \alpha_1 + \ldots + \alpha_s = M. \qquad (6.26)$$

Note that we know $C_{(2,2)}$ and $C_{(3)}$. Suppose vertex i in G has degree d_i. Then from the Mayer expansion we have,

$$C_{(2,2)} = -\sum_i \binom{d_i}{2} \text{ and } C_{(3)} = \sum_i \binom{d_i}{2}. \qquad (6.27)$$

Finally, in the proof of theorem 6.2 we use corollary 6.6 to show that when q is large enough (as stated in the theorems) the first term of the Mayer expansion dominates and thus implies log-concavity as shown in 6.2.3. These steps together complete the proofs of theorems 1.2 and 1.3

6.2.1 Proof of theorem 6.4

In this section we will prove theorem 6.4. We will need the following theorems due to A. Sokal [30] and C.Borgs [5]. First we explain some notation.

Let X be a single particle state space with relation $\sim$ on $X \times X$ and and $w : X \to \mathbb{C}$ a complex function called the *fugacity vector*.

We say $X' \subseteq X$ is *independent* if $x \sim y$ for all $x, y \in X'$.
Let,

$$Z_X(w) = \sum_{\substack{X' \subseteq X \\ X' \text{ independent}}} \prod_{x \in X'} w_x. \tag{6.28}$$

Theorem 6.7 (Dobrushin's theorem as stated in [5])**.** *In the above setup Z_X is non-zero in the region $|w_x| \leq R_x$, if there exists constants $c_x \geq 0$ such that,*

$$R_x \leq (e^{c_x} - 1) \exp\left(-\sum_{y \nsim x} c_y\right). \tag{6.29}$$

Further,

$$\left|\log\left\{\frac{Z_X}{Z_{X'}}\right\}\right| \leq \sum_{x \in X - X'} c_x, \text{ for all } X' \subseteq X. \tag{6.30}$$

Hence, in particular,

$$|\log Z_X| \leq \sum_{x \in X} c_x. \tag{6.31}$$

From Dobrushin's theorem follows the Kotecky-Preiss condition:

Theorem 6.8 (Kotecky-Preiss condition)**.** *In the above setup Z_X is non-zero in the region $|w_x| \leq R_x$, if there exists constants $c_x \geq 0$ such that,*

$$R_x \leq c_x \exp\left(-\sum_{y \nsim x} c_y\right) \tag{6.32}$$

We will use the following consequence of the Kotecky-Preiss condition as stated by Sokal [30],

Theorem 6.9 (Proposition 3.2 of [30])**.** *Let $R_x \geq 0$ for all $x \in X$. Suppose that $X = \bigcup_{n=1}^{\infty} X_n$ is a disjoint union such that there exist constants $\{A_n\}_{n=1}^{\infty}$ and α such that,*

1. $\sum_{y \in X_n : y \nsim x} R_y \leq A_n m$, *for all m, n and all $x \in X_m$.*

2. $\sum_{n=1}^{\infty} e^{\alpha n} A_n \leq \alpha.$

Then the Kotecky-Preiss condition holds with the choice $c_x = e^{\alpha n} R_x$ for all $x \in X_n$.

Corollary 6.10. *Assume the hypothesis of theorem 6.9. Further let $F \subseteq X_2$ be such that for all $y \in X_n$ there is a $v \in F$ such that $y \nsim v$. Then,*

$$|\log Z_X| \leq \sum_{x \in X} c_x \leq |F|\alpha. \tag{6.33}$$

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Proof. By choosing $m = 1$ in part 1 of 6.9 we have,

$$\sum_{y \in X_n : y \nsim v} e^{\alpha n} R_y \leq 2 e^{\alpha n} A_n \text{ for all } v \in F. \tag{6.34}$$

Thus,

$$\sum_{x \in X} c_x \leq \sum_{n \geq 1} \sum_{v \in F} \sum_{\substack{y \in X_n \\ y \nsim z}} e^{\alpha n} R_y \leq \sum_{n \geq 1} \sum_{v \in F} 2 e^{\alpha n} A_n \leq 2|F|\alpha. \tag{6.35}$$

$\square$

Theorem 6.11 (Penrose theorem [27])**.** *Let $G = (V, E)$ be a finite graph on n vertices. Then,*

$$\left| \sum_{\substack{E' \subseteq E \\ (V, E') connected}} (-1)^{|E'|} \right| \leq T_n(G), \tag{6.36}$$

where $T_n(G)$ denotes the number of spanning trees of G.

Theorem 6.12 (A. Sokal, [30])**.** *Let $G = (V, E)$ be a graph on n vertices with maximum degree Δ. Then,*

$$T_n(G) \leq t_n^{\Delta}, \tag{6.37}$$

where,

$$t_n^{\Delta} = \Delta \frac{[(\Delta - 1)(n + 1)]!}{n![(\Delta - 2)n + \Delta]!}. \tag{6.38}$$

Theorem 6.13 (A.Sokal [30])**.** *Let Q be the smallest number such that,*

$$\inf_{\alpha > 0} \frac{1}{\alpha} \sum_{n=2}^{\infty} e^{\alpha n} Q^{-(n-1)} t_n^{(r)} \leq 1. \tag{6.39}$$

Then the choice $\alpha = 2/5$ and $Q = K\Delta = 7.963907\Delta$ satisfies the above inequality. Hence it follows that $Q \leq K\Delta = 7.963907\Delta$.

Proof of Theorem 6.4

Proof. Let $G = (V, E)$ be a graph of maximum degree Δ. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote the graph whose set of vertices is given by set of subsets S of V such that $|S| \geq 2$ and there is an edge between S_1 and S_2 if $S_1 \cap S_2 \neq \emptyset$. Let X_i denote the set of connected subsets of V of size i. Now we apply the above theorem for $X = \mathcal{V} = \bigsqcup_{i=2}^{|V|} X_i$ and relation $x \sim y$ denoting that x, y are disjoint in $\mathcal{G}$.

The generalized chromatic polynomial can be written as follows:

$$P(z_1, \ldots, z_q, G) = \sum_{\substack{\mathcal{W} \subseteq \mathcal{V} \\ \mathcal{W} \text{ independent}}} \prod_{S_i \in S} w(S_i)$$

where $w(S) = (z_1^{|S|} + \ldots + z_q^{|S|}) \sum_{\substack{\gamma \subseteq E, \\ (S, \gamma) \text{ connected}}} (-1)^{|\gamma|}.$ \tag{6.40}

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Now we will imitate the proof of Theorem 5.1 of [30]. We apply theorem 6.9 with the choices,

$$R_S = |w(S)|, \tag{6.41}$$

and,

$$A_n = \max_{x \in V} \sum_{\substack{x \in S \\ |S| = n}} |w(S)|. \tag{6.42}$$

By the definition of Ω we have,

$$|z_1^{|S|} + \ldots + z_q^{|S|}| \leq \left(2\sqrt{\frac{\Delta}{q}}\right)^{|S|-1}. \tag{6.43}$$

Thus,

$$|w(S)| \leq \left(2\sqrt{\frac{\Delta}{q}}\right)^{|S|-1} \sum_{\substack{\gamma \subseteq E, \\ (S,\gamma) \text{ connected}}} (-1)^{|\gamma|}$$

$$\left(2\sqrt{\frac{\Delta}{q}}\right)^{|S|-1} T_n((S,E)) \leq \left(2\sqrt{\frac{\Delta}{q}}\right)^{|S|-1} t_n^{(\Delta)} \tag{6.44}$$

Let Q be the smallest number such that,

$$\inf_{\alpha > 0} \frac{1}{\alpha} \sum_{n=2}^{\infty} e^{\alpha n} Q^{-(n-1)} t_n^{(\Delta)} \leq 1. \tag{6.45}$$

Then $P \neq 0$ if $\left(\sqrt{\frac{q}{4\Delta}}\right)^{|S|-1} > Q$. By theorem 6.13 we have $Q \leq K\Delta$. So it suffices to have $q > 4K^2\Delta^3$.

For theorem 6.3 we have $|z_1^{|S|} + \ldots + z_q^{|S|}| \leq \frac{1}{q-1}$. Hence it suffices to have $q > 1 + K\Delta$.

Thus, $P_G \neq 0$ in the region $\{(z_1, \ldots, z_q) : |z_1|^m + \ldots + |z_q|^m \leq \left(2\sqrt{\frac{\Delta}{q}}\right)^{m-1} \; \forall m \in \mathbb{Z}_+\}$ when $q > 4K^2\Delta^3$ and $P_G \neq 0$ in the region $\{(z_1, \ldots, z_q) : |z_1|^m + \ldots + |z_q|^m \leq \left(\frac{1}{q-1}\right)^{m-1} \; \forall m \in \mathbb{Z}_+\}$ when $q > K\Delta$.

Further, by corollary 6.10 (with F being the set of edges in G) and theorem 6.13 we also have in both cases that

$$|\log P_G(p_1, \ldots, p_q)| \leq 2|F|\alpha = 4|E|/5. \tag{6.46}$$

$\square$

6.2.2 Proof of corollary 6.6

Proof. In this section we prove corollary 6.6

Let, $\mathbf{r} = (r_1, \ldots r_s)$ be a vector for $r_i \geq 0$. Define,

$$\mathcal{M}_r(f(\nu(\alpha_1), \ldots, \nu(\alpha_s))) = \frac{1}{(2\pi)^s} \int_{\{(\theta_1, \ldots, \theta_s): 0 \leq |\theta_i| \leq 2\pi\}} f(r_1 e^{i\theta_1}, \ldots, r_s e^{i\theta_s}) d\theta_1 \ldots d\theta_s. \quad (6.47)$$

Note that,

$$\mathcal{M}_r\left(\frac{\nu(\beta_1) \ldots \nu(\beta_t)}{\nu(\alpha_1) \ldots \nu(\alpha_s)}\right) = 0, \text{ for } \beta \neq \alpha. \quad (6.48)$$

Hence,

$$\mathcal{M}_r\left(\frac{\log P_G}{\nu(\alpha_1) \ldots \nu(\alpha_s)}\right) = C_\alpha. \quad (6.49)$$

Also,

$$\mathcal{M}_r\left(\frac{\log P_G}{\prod_1^s \nu(\alpha_i)}\right) \leq \frac{4|E|}{5|\prod_1^s \nu(\alpha_i)|} \leq \frac{4|E|}{5\prod_{i \leq s} r_i} \quad (6.50)$$

By theorem 6.4 we know that $\log P_G$ converges when $q \geq K^2 \Delta^3$ and $\nu(\alpha_i) \leq l(q)^{\alpha_i - 1}$.

Thus, the above inequality holds for $q = K^2 \Delta^3$ and $\nu(\alpha_i) \leq l(q)^{\alpha_i - 1}$. In this case $l(q) = \frac{1}{K\Delta}$. Hence, using $r_i = \nu(\alpha_i)$ we get,

$$C_\alpha \leq \frac{4|E|}{5\prod_{i \leq s} l(q)^{\alpha_1 - 1}}. \quad (6.51)$$

Thus,

$$C_\alpha \leq 4|E|(K\Delta)^{(M-s)}/5, \text{ for } \alpha = (\alpha_1 \leq \ldots \leq \alpha_s) \text{ a partition of} M. \quad (6.52)$$

$\square$

6.2.3   Proof of theorem 6.2

We need a small lemma before we complete the proof.

Lemma 6.14. *Let* $\Theta = \{(a_1, \ldots, a_q) : \sum a_i = 1\}$. *If*

$$g(a_1, \ldots, a_q) = f(a_1, \ldots, a_q) - (a_1^{s+r} + \ldots + a_q^{s+r})$$

is minimized on Θ *at* $(1/q, \ldots, 1/q)$ *then so is*

$$h(a_1, \ldots, a_q) = f(a_1, \ldots, a_q) - (a_1^{s+1} + \ldots + a_q^{s+1})(a_1^r + \ldots + a_q^r).$$

Proof. Note that,

$$h(a_1, \ldots, a_q) = g(a_1, \ldots, a_q) - (a_1^{s+1} + \ldots + a_q^{s+1})(a_1^r + \ldots + a_q^r) + (a_1^{s+r} + \ldots + a_q^{s+r}).$$

Now, since g is minimized at $(1/q, \ldots, 1/q)$, it suffices to prove that

$$w(a_1, \ldots, a_q) = -(a_1^{s+1} + \ldots + a_q^{s+1})(a_1^r + \ldots + a_q^r) + (a_1^{s+r} + \ldots + a_q^{s+r})$$

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is minimized at $(1/q, \ldots, 1/q)$. This is true since $w(1/q, \ldots, 1/q) = 0$ and in general $w(a_1, \ldots, a_q) \geq 0$. To see this, note that,

$$
\begin{aligned}
w(a_1, \ldots, a_q) = &- (a_1^{s+1} + \ldots + a_q^{s+1})(a_1^r + \ldots + a_q^r) \\
&+ (a_1^{s+r} + \ldots + a_q^{s+r})(a_1 + \ldots + a_q) \\
= &\sum_{i \neq j} (a_i^1 a_j^{s+r} + a_j^1 a_i^{s+r} - a_i^{s+1} a_j^r - a_j^{s+1} a_i^r) \geq 0 \text{ by AM-GM} .
\end{aligned}
\tag{6.53}
$$

This completes the proof.

$\square$

Now we complete the proof of theorem 6.2.

Proof. As observed above,

$$
\log P_G(p_1, \ldots, p_q) = -|E|(p_1^2 + \ldots + p_q^2) + \sum_i \binom{d_i}{2}(p_1^3 + \ldots, p_q^3) - \sum_i \binom{d_i}{2}(p_1^2 + \ldots, p_q^2)^2
$$

$$
+ \sum_{M=5}^{\infty} \sum_{\substack{\alpha=(\alpha_1 \leq \ldots \leq \alpha_s): \\ \text{partition of } M \\ \alpha_i \geq 2}} C_\alpha \prod_{i \leq s} (p_1^{\alpha_i} + \ldots + p_q^{\alpha_i})
$$

$$
\tag{6.54}
$$

and,

$$
C_\alpha \leq 4|E|(K\Delta)^M/5, \text{ for } \alpha \text{ a partition of} M.
\tag{6.55}
$$

Now, by theorem 6.14 it suffices to show that $\tilde{P}_G(p_1, \ldots, p_q)$ is maximized when $p_1 = \ldots = p_q$, where,

$$
\tilde{P}_G(p_1, \ldots, p_q) = -|E|(p_1^2 + \ldots + p_q^2) + 2\sum_i \binom{d_i}{2}(p_1^3 + \ldots, p_q^3)
$$

$$
+ \sum_{M=5}^{\infty} \sum_{\substack{\alpha=(\alpha_1 \leq \ldots \leq \alpha_s): \\ \text{partition of } M \\ \alpha_i \geq 2}} B|V|(K\Delta)^M (p_1^{M-s+1} + \ldots p_q^{M-s+1})
$$

$$
\tag{6.56}
$$

$$
= +2\sum_i \binom{d_i}{2}(p_1^3 + \ldots, p_q^3)
$$

$$
- |E|(p_1^2 + \ldots + p_q^2) + \sum_{k=3}^{\infty} A(k) \times \frac{4|E|}{5}(K\Delta)^k (p_1^{k+1} + \ldots p_q^{k+1}),
$$

where, $A(k)$ denotes the number of ordered partitions of k into exactly k parts. The second equality follows since for every partition $\alpha = (\alpha_1 \leq \ldots \leq \alpha_s)$ of M such that $\alpha_i \geq 2$, we get a unique $\beta = (\alpha_1 - 1 \leq \ldots \alpha_q - 1)$, a partition of $M - s$. Note, $A(k) \leq 2^k$. The Hessian of $\tilde{P}_G(p_1, \ldots, p_q)$ is a diagonal matrix with i'th diagonal entry given by,

$$
H_{ii} = -|E| + 12\sum_i \binom{d_i}{2} p_i + \frac{4|E|}{5} \sum_{k=3}^{\infty} A(k) \times (K\Delta)^k k(k+1) p_i^{k-1}.
\tag{6.57}
$$

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Since $\sum_i d_i = |E|$ and $d_i \leq \Delta$, we have,

$$\sum_i \binom{d_i}{2} \leq \frac{1}{2} \sum_i d_i^2 \leq |E|\Delta. \text{ since } x^2 \text{ is a convex function.} \tag{6.58}$$

Using above inequality and $A(k) \leq 2^k$ gives,

$$H_{ii} \leq -|E| + \frac{4|E|}{5} \sum_{k=3}^{\infty} 2^k k(k+1)(K\Delta)^k p_i^{k-1}$$
$$\leq -|E| \left(\frac{1}{2} - 12\Delta p_i + \sum_{k\geq 3} \frac{1}{2^{k-1}} - (4/5)2^k k(k+1)(K\Delta)^k p_i^{k-1}/5 \right) \tag{6.59}$$

In the case of theorem 1.2 we have $p_i \leq \left(\frac{4\Delta}{q} \right)^{1/2}$ thus,

$$H_{ii} \leq -|E| \left(\frac{1}{2} - 12\Delta \left(\frac{4\Delta}{q} \right)^{1/2} + \sum_{k\geq 3} \frac{1}{2^{k-1}} - (4/5)2^k k(k+1)(K\Delta)^k \left(\frac{4\Delta}{q} \right)^{\frac{k-1}{2}} \right) \tag{6.60}$$

In this case $H_{ii} < 0$ if $q > 6.3 \times 10^5 \Delta^4$. $H_{ii} < 0$ implies that P is log-concave. This completes the proof of 1.2.

In the case of 1.3 we have $p_i \leq \frac{1}{q-1}$. Thus,

$$H_{ii} \leq -|E| \left(\frac{1}{2} - 12\Delta \frac{1}{q-1} + \sum_{k\geq 3} \frac{1}{2^{k-1}} - (4/5)2^k k(k+1)(K\Delta)^k \left(\frac{1}{q-1} \right)^{k-1} \right). \tag{6.61}$$

In this case $H_{ii} < 0$ if $q > 400\Delta^{3/2}$. $H_{ii} < 0$ implies that P is log-concave. This completes the proof of 1.3.

$\square$

7 Complete graphs and cycles are P-uniform

In this section we show that complete graphs and cycles are P-uniform. As a quick reminder of the definitions: Let $P_G(k, p_1, \ldots, p_q)$ denote the probability that at most k edges are monochromatic when vertices of G are independently assigned colors according to distribution $p = (p_1, \ldots, p_q)$. We considered the case $k = 0$ above. We will say that a graph G is 'P-uniform' if for all q and k, probability $P_G(k, p_1, \ldots, p_q)$ is maximized by the uniform distribution. As shown in Section 1, the graph in Figure 1 is claw-free but not P-uniform.

Theorem 7.1. *The complete graph G_n on n vertices is P-uniform.*

Proof. First we prove the above statement for $q = 2$. The probability that t vertices of K_n are colored with one color and the remaining vertices with the other color is,

$$S_t = (p_1^t p_2^{n-t} + p_2^t p_1^{n-t}) \tag{7.1}$$

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In this case the number of monochromatic edges is,

$$\binom{t}{2} + \binom{n-t}{2} = \frac{t^2 + (n-t)^2}{2} - \frac{n}{2}. \tag{7.2}$$

This is a decreasing function of t as t goes from 0 to $\lfloor n/2 \rfloor$.

Thus, the probability of at most k badly colored edges is,

$$A_k = \sum \binom{n}{t} S_t \tag{7.3}$$

where t runs over the non-negative integers satisfying $\binom{t}{2} + \binom{n-t}{2} \le k$.

Let,

$$B_m(p_1, p_2) = \sum_{t=m}^{\lfloor n/2 \rfloor} \binom{n}{t} S_t. \tag{7.4}$$

We want to show that A_k is maximized when $p_1 = p_2 = 1/2$ for all k. Equivalently it is enough to prove that $B_m(p_1, p_2)$ is maximized when $p_1 = p_2 = 1/2$ for all m.

Suppose this is not true. Let m be the largest integer such that $B_m(p_1, p_2)$ is not maximized when $p_1 = p_2 = 1/2$. Then S_m is also not maximized when $p_1 = p_2 = 1/2$. Suppose it is instead maximized for $p_1' \ne p_2'$.

Now observe that

$$S_{t+1} \le S_t \text{ for all } 0 \le t \le \lfloor n/2 \rfloor$$

with equality when $p_1 = p_2 = 1/2$.

So,

$$S_m(p_1', p_2') > S_m(1/2, 1/2) \Rightarrow S_u(p_1', p_2') > S_u(1/2, 1/2) \tag{7.5}$$

for all $0 \le u \le m$. Hence,

$$B_1(p_1', p_2') = B_m(p_1', p_2') + \sum_{t=1}^{m-1} S_t(p_1', p_2')$$

$$> B_m(1/2, 1/2) + \sum_{t=1}^{m-1} S_t(1/2, 1/2). \tag{7.6}$$

But,

$$B_1(1/2, 1/2) = (p_1 + p_2)^n - (p_1^n + p_2^n) = 1 - (p_1^n + p_2^n) \tag{7.7}$$

which is maximized when $p_1 = p_2 = 1/2$. Thus we have a contradiction. Hence $B_m(p_1, p_2)$ is maximized when $p_1 = p_2 = 1/2$ for all $m \ge 0$. This completes the case of two colors.

Now we proceed to prove the statement for $q > 2$ colors. The vertices are colored randomly with color c_i having probability p_i.
Suppose a_i vertices are colored with color i for $i \ge 3$. The probability of this is

$$\binom{n}{a_3 \dots a_q} \prod_{i=3}^{q} p_i^{a_i}. \tag{7.8}$$

We are looking for the probability that the number of monochromatic edges is less than or equal to k for a constant k. Let $P_{a_3 \ldots a_q}$ denote the probability that exactly a_i vertices are colored with color c_i for $i \geq 3$ and less than or equal to k edges are monochromatic. Then, $P_{a_3 \ldots a_q}$ equals the probability that exactly a_i vertices are colored with color c_i for $i \geq 3$ times probability that the remaining $h = n - (a_3 + \ldots + a_r)$ vertices are colored with colors c_1, c_2 such that at most $k - \sum_{i=3}^q \binom{a_i}{2}$ edges are monochromatic. i.e.

$$P_{a_3 \ldots a_q} = \begin{cases} 0 & \text{if } \sum_{i=3}^q \binom{a_i}{2} > k \\ \binom{n}{a_3 \ldots a_q} \prod_{i=3}^q p_i^{a_i} A_h & \text{if } \sum_{i=3}^q \binom{a_i}{2} \leq k. \end{cases} \tag{7.9}$$

For fixed a_i this is maximized when $p_1 = p_2$ by our result for $q = 2$.

Thus,

$$\sum_{\sum_{i=3}^q a_i \leq n} P_{a_3 \ldots a_q}. \tag{7.10}$$

Each of the summands is individually maximized when $p_1 = p_2$. Hence the total is also maximized for $p_1 = p_2$.

If $P_k(p_1 \ldots p_q)$ denotes the probability that a random coloring of the vertices with each color c_i occurring with probability p_i then the above tells us

$$P_k(p_1 \ldots p_q) \leq P_k(\frac{p_1 + p_2}{2}, \frac{p_1 + p_2}{2}, p_3 \ldots p_q) \tag{7.11}$$

and by symmetry

$$P_k(p_1 \ldots p_q) \leq P_k(p_1 \ldots p_{i-1} \frac{p_i + p_j}{2}, p_{i+1} \ldots p_{j-1}, \frac{p_i + p_j}{2}, p_{j+1} \ldots p_q) \tag{7.12}$$

with equality only if $p_j = p_i$.

Since $P_k : [0,1]^q \to \mathbb{R}$ is a continuous function on a compact space, a maximum is attained and by the above inequality it must be attained when $p_1 = \cdots = p_q = 1/q$.
This completes the proof.

$\square$

Remarks: 1) In [22] Lars Holst considers a very similar problem. He shows that the probability of there being a bounded number of vertices of the same color is also a Schur concave function. His proof also reduces to showing that 7.3 is Schur concave which he shows using an interesting calculus proof.
2) Geir Helleloid also has a combinatorial proof of the above theorem.

Next we prove that chains are also strongly concave. First we define *concave monotonicity* and prove a lemma relating it to strong concavity.

Definition: Let G be a graph on n vertices. For $s \leq \frac{n}{2}$ we call a coloring of G with two colors an s-coloring of G if the coloring divides the set of vertices into two sets of size s and $n - s$ each colored with a distinct color. Let $G(s, c)$ denote the proportion of s-colorings of G with at most

c monochromatic edges. Then we say that G is *concave-monotone* if $G(s_1, c) \leq G(s_2, c)$ for all $s_1 \leq s_2$ and all c.

Lemma 7.2. *If G is concave-monotone then G is also P-uniform.*

Proof. The probability that a random coloring of the vertices with two colors has at most c monochromatic edges is:

$$
\begin{aligned}
D_k &= \sum_{t=0}^{\lfloor n/2 \rfloor} G(t, c) \binom{n}{t} S_t + G(0, c) B_0(p_1, p_2) \\
&+ \sum_{m=1}^{\lfloor n/2 \rfloor} (G(m, c) - G(m-1, c)) \binom{n}{m} B_m(p_1, p_2)
\end{aligned}
\tag{7.13}
$$

which is maximized for $p_1 = p_2 = 1/2$ since $(G(m, c) - G(m-1, c)) \geq 0$ for all m, $G(0, c) \geq 0$ and $B_m(p_1, p_2)$ is also maximized for $p_1 = p_2 = 1/2$ for all m as seen in the proof of 7.1. Hence the lemma follows. $\square$

Theorem 7.3. *Chains are concave-monotone and hence P-uniform.*

Proof. Suppose we use s red and $t \geq s$ blue colors to color a chain C_n of n vertices. We calculate the number of ways to color the chain so that there are exactly $n - 1 - r$ monochromatic edges i.e. there are r pairs of neighboring vertices of different color. If r is even (odd) then the first and last vertices of the chain must have the same (different) color. We denote the number of s- colorings of C_n that have exactly $n - r - 1$ monochromatic edges by $N(s, r)$. Thus,

$$
\begin{aligned}
N(s, r) = &\binom{s-1}{h}\binom{t-1}{h-1} + \binom{t-1}{h}\binom{s-1}{h-1} \text{ if } r = 2h < 2s \\
&\binom{t-1}{h} \text{ if } r = 2h = 2s, s < t \\
&0 \text{ if } r = 2s, s = t.
\end{aligned}
\tag{7.14}
$$

$$
2\binom{s-1}{h}\binom{t-1}{h} \text{ if } r = 2h + 1
$$

The chain is convex monotone if

$$
\frac{\sum_{r=c}^{n-1} N(s, r)}{\binom{s+t}{s}} \geq \frac{\sum_{r=c}^{n-1} N(s-1, r)}{\binom{s+t}{s-1}}.
\tag{7.15}
$$

or equivalently,

$$
\sum_{r=c}^{n-1} \frac{N(s, r)}{\binom{s+t}{s}} - \frac{N(s-1, r)}{\binom{s+t}{s-1}} \geq 0.
\tag{7.16}
$$

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Note,

$$\sum_{r=0}^{n-1} \frac{N(s,r)}{\binom{s+t}{s}} - \frac{N(s-1,r)}{\binom{s+t}{s-1}} = 0. \tag{7.17}$$

We will now analyze

$$\beta(s,r) = \frac{N(s,r)}{\binom{s+t}{s}} - \frac{N(s-1,r)}{\binom{s+t}{s-1}}. \tag{7.18}$$

Since $s \leq t$, it follows that the maximum number of non-monochromatic edges possible is $2s$.

And for $r = 2h$ we have,

$$\begin{aligned}
\beta(s,r) &= \frac{N(s,r)}{\binom{s+t}{s}} - \frac{N(s-1,r)}{\binom{s+t}{s-1}} \\
&= \frac{\binom{s-1}{h}\binom{t-1}{h-1} + \binom{t-1}{h}\binom{s-1}{h-1}}{\binom{s+t}{s}} - \frac{\binom{s-2}{h}\binom{t}{h-1} + \binom{t}{h}\binom{s-2}{h-1}}{\binom{s+t}{s-1}} \\
&= \alpha(s,r)\Big(\frac{(s-1)s}{(s-1-h)(t-h)} + \frac{s(s-1)}{(s-h)(s-h-1)} - \frac{(t+1)t}{(t+1-h)(t-h)} - \frac{t(t+1)}{(t-h)(s-h-1)}\Big) \\
&= \alpha(s,r)(t+s-2h)(t+1-s)(ht+hs-st-s).
\end{aligned} \tag{7.19}$$

where

$$\alpha(s,r) = \frac{(s-2)!(t-1)!(s-1)!t!}{h!(h-1)!(s+t)!(s-h)!(t+1-h)!} > 0.$$

And for $r = 2h+1$ we have,

$$\begin{aligned}
\beta(s,r) &= \frac{N(s,r)}{\binom{s+t}{s}} - \frac{N(s-1,r)}{\binom{s+t}{s-1}} \\
&= 2\frac{\binom{s-1}{h}\binom{t-1}{h}}{\binom{s+t}{s}} - 2\frac{\binom{s-2}{h}\binom{t}{h}}{\binom{s+t}{s-1}} \\
&= 2\nu(s,r)(t+1-s)(s+t-2h)(t+sh+ht-st).
\end{aligned} \tag{7.20}$$

where

$$\nu(s,r) = \frac{(s-2)!(t-1)!(s-1)!t!}{h!h!(s+t)!(s-1-h)!(t-h)!} > 0.$$

It follows from 7.19 and 7.20 that $\beta(s,r) \leq 0$ for all $h \leq \frac{t(s-1)}{(s+t)}$ and $\beta(s,r) \geq 0$ for all $h \geq \frac{(t+1)s}{(s+t)} = \frac{t(s-1)}{(s+t)} + 1$. This together with 7.17 proves 7.16 and hence completes the proof. $\qquad\square$

Remarks:

- Theorem 7.1 could also be derived from Lemma 7.2.

- Not all claw-free graphs are concave-monotone. Apart from complete graphs, cycles and chains we have not found other interesting examples of families of concave-monotone graphs.

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8   Acknowledgements

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